Czechoslovak Mathematical Journal, 52 (127) (2002), 225–235

EXTREMAL METRICS AND MODULUS

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(Received March 31, 1998)

Abstract. We give a new proof of Beurling's result related to the equality of the extremal length and the Dirichlet integral of solution of a mixed Dirichlet-Neuman problem.

Our approach is influenced by Gehring's work in $\mathbb{R}^3$ space. Also, some generalizations of Gehring's result are presented.

Keywords: extremal distance, conformal capacity, Beurling theorem

MSC 2000: 30A15, 30C85

Introduction

Beurling proved the following result (see Ahlfors [1]):

Theorem 0.1. (Beurling's theorem) *Let Ω be a region in the complex plane bounded by a finite number of analytic Jordan curves, let E_0 and E_1 be disjoint and consist of a finite number of closed arcs or curves in the boundary of Ω. Then the extremal distance $d_\Omega(E_0, E_1)$ is the reciprocal of the Dirichlet integral*

$$D(u) = \iint_\Omega (u_x^2 + u_y^2)\,\mathrm{d}x\,\mathrm{d}y,$$

where u satisfies

(i) *u is bounded and harmonic in Ω,*

(ii) *u has a continous extension to $\Omega \cup E_0^\circ \cup E_1^\circ$, and $u = 0$ on E_0 and $u = 1$ on E_1,*

(iii) *the normal derivative $\frac{\partial u}{\partial n}$ exists and vanishes on $C_\circ$ (C denotes the full boundary of Ω, $C_\circ = C - (E_0 \cup E_1)$, and E_0° and E_1° denote the relative interiors of E_0 and E_1 as subsets of C).*

The proof is based on two important ingredients:

1) existence of a solution of a mixed Dirichlet-Neuman problem (we denote it by u),

2) decomposition of the domain to rings and quadrilateral subdomains using, in fact, the orthogonal and vertical trajectories of the quadratic differential defined by u.

For the theory of trajectories of holomorphic quadratic differentials see Gardiner [7] and Strebel [5].

Our first purpose was to give a more elementary proof of this result (that is, with no use of these two subjects), using a minimizing sequence (see for example Courant's book [6]), and to derive some equalities not contained in the proof of Beurling's theorem.

During our work on this problem we became aware of Gehring's papers ([2], [3]), which strongly influenced our research.

In [2] and [3] Gehring proved that Väisälä's definition of extremal distance between E_0 and E_1 in Ω (see [9]) is essentially equivalent to Dirichlet's integral definition due to Loewner (see [10]) if Ω is a ring domain in $\mathbb{R}^3$, and E_0 and E_1 are boundary components of Ω (cf. also [4]). Gehring used this result to study quasiconformal mappings in space.

We generalize this result to the setting of smooth domains in $\mathbb{R}^n$. An application of this result gives a short proof of Beurling's Theorem.

As we understand, there are additional technical difficulties if we work with general domains instead of ring domains. Because of that, we need Lemma 2.1.

1. Notation

Definition 1.1. Let Ω be an open set in $\mathbb{R}^n$ and Γ a set whose elements γ are rectifiable arcs in Ω. Let ϱ be a nonnegative Borel measurable function in Ω (such ϱ we will call a metric). We can define the *ϱ-length* of γ by

$$L(\gamma,\varrho)=\int_\gamma \varrho\,|\mathrm{d}x|,$$

the *ϱ-volume* of Ω as

$$V(\Omega,\varrho)=\int_\Omega \varrho^n\,\mathrm{d}V(x),$$

where $\mathrm{d}V$ is the n-dimensional Lebesgue measure in $\mathbb{R}^n$, and the *minimum length* of Γ by $L(\Gamma,\varrho)=\inf\limits_{\gamma\in\Gamma} L(\gamma,\varrho)$. The *modulus* of Γ in Ω is defined by $\operatorname{mod}_\Omega(\Gamma)=\inf\limits_{\varrho}\frac{V(\Omega,\varrho)}{L(\Gamma,\varrho)^n}$ where ϱ is subject to the condition $0<V(\Omega,\varrho)<\infty$. The *extremal length* of Γ in Ω is defined as $\Lambda_\Omega(\Gamma)=\operatorname{mod}_\Omega(\Gamma)^{\frac{1}{1-n}}$.

Definition 1.2. Let Ω be an open set in $\mathbb{R}^n$, and let E_0, E_1 be two sets in the closure of Ω. Take Γ to be the set of all connected arcs in Ω which join E_0 and E_1, i.e. each $\gamma \in \Gamma$ has one endpoint in E_0 and one in E_1, and all other points of γ are in Ω. The extremal length $\Lambda(\Gamma)$ is called the *extremal distance* of E_0 and E_1 in Ω, and we denote it by $d_\Omega(E_0, E_1)$.

Now, let Ω be a bounded region whose boundary consists of a finite number of C^1 hypersurfaces, and E_0, E_1 are disjoint, and each is a finite union of closed hypersurfaces contained in the boundary of Ω. Then we define the *conformal n-capacity* of Ω as

$$C[\Omega, E_0, E_1] = \inf_u \int_\Omega |\nabla u|^n \,\mathrm{d}V(x),$$

where the infimum is taken over all functions $u\colon\ \Omega \to \mathbb{R}$ which are differentiable in Ω, continuous in $\overline{\Omega}$ and have boundary values 0 on E_0 and 1 on E_1.

From now on let Γ be the family of arcs in Ω which join E_0 and E_1.

Definition 1.3. If u is continuous and ACL in Ω, and u has boundary values 0 on E_0 and 1 on E_1, we say that u is an admissible function for the domain Ω with respect to E_0 and E_1 and denote it by $u \in \mathrm{E}(\Omega, E_0, E_1)$.

2. Extremal distance and conformal capacity

In this section we want to prove that

$$d_\Omega(E_0, E_1) = C[\Omega, E_0, E_1]^{\frac{1}{1-n}}.$$

Lemma 2.1. *Let f be a metric in Ω and $V(\Omega, f) < \infty$. Then there exists a neighborhood U of $\partial\Omega$, a metric $\tilde{f}$ on U, and a diffeomorphism A of U onto itself such that*

i) *$\tilde{f} = f$ on $U \bigcap \Omega = U'$,*

ii) *A is the identity on $\partial\Omega$ and $A(U') = U''$, where $U'' = U \bigcap \Omega^c$,*

iii) *for every rectifiable curve γ in U'' we have*

$$L(\gamma, \tilde{f}) \geqslant L(A(\gamma), f),$$

iv) *$V(U'', \tilde{f}) \leqslant K\, V(U', f)$, where K is a finite constant,*

v) *K and U do not depend on f.*

P r o o f. The Tubular theorem (see [8]) yields that there exists a neighborhood U of $\partial\Omega$ such that there exists a diffeomorphism H from U onto $(-1,1)\times\partial\Omega$ and $H(x)=(0,x)$ for $x\in\partial\Omega$. For U small enough, we have for the Jacobian J_H of H that $0<m<|J_H|<M<\infty$. Let S be the mapping from $(-1,1)\times\partial\Omega$ onto itself defined as $S((t,x))=(-t,x)$. Define A as $A=H^{-1}\circ S\circ H$. We obtain that $A\circ A=\mathrm{id}$ and $A(U')=U''$. For the Jacobian J_A of A we have $\frac{m}{M}<|J_A|<\frac{M}{m}$, and it folows that $|A'(x)|^n\leqslant K\,|J_A(x)|$ for some $K<+\infty$.

Let now x be from U''. Define $\tilde f(x)$ as $\tilde f(x)=f(A(x))\,|A'(x)|$. Then for a rectifiable curve γ in U'' we have

$$\int_\gamma \tilde f(x)\,|\mathrm{d}x| = \int_\gamma f(A(x))\,|A'(x)|\,|\mathrm{d}x| \geqslant \int_{A(\gamma)} f(y)\,|\mathrm{d}y|.$$

We also conclude that

$$\begin{aligned}\int_{U''}\tilde f^n(x)\,\mathrm{d}V(x) &= \int_{U''} f^n(A(x))\,|A'(x)|^n\,\mathrm{d}V(x)\\ &\leqslant K\int_{U''} f^n(A(x))\,|J_A(x)|\,\mathrm{d}V(x) = K\int_{U'} f^n(y)\,\mathrm{d}V(y).\end{aligned}$$

□

From now on, we suppose that any metric f is defined in some neighborhood of the domain $\overline{\Omega}$ (namely, $\Omega^*=\Omega\bigcup U$), and that we have a diffeomorpfism A of each outside boundary strip small enough onto an appropriate inside boundary strip.

Lemma 2.2. *Let S_r be a spherical surface of radius r, and let f be a metric on S_r. Then each pair of points P and Q on S_r can be joined by a circular arc $\alpha\subset S_r$ such that*

$$\left(\int_\alpha f(x)\,|\mathrm{d}x|\right)^n \leqslant A\,r\int_{S_r} f^n(x)\,\mathrm{d}\sigma_r(x),$$

where $\mathrm{d}\sigma_r$ is the Lebesgue measure on S_r and A is a constant depending only on n.

P r o o f. Let $d(P,Q)=\inf\limits_\beta(L(\beta,f))$, where infimum is taken over all circular arcs on S_r which join the points P and Q. We suppose that this infimum is positive (the case when it is zero is left to the reader). Then there exists a circular arc α such that $L(\alpha,f)\leqslant 2\,d(P,Q)$.

Without loss of generality, we can assume that $r=1$ and $P=(0,0,\dots,0,1)$, and denote $\mathbb{S}_1$ by $\mathbb{S}$.

Now we map $\mathbb{S}$ stereographically by p onto $Z=\mathbb{R}^{n-1}$. Then P corresponds to ∞, Q to some point a, and hence we obtain

$$d(P,Q)\leqslant L(\beta,f)=\int_\beta f(x)\,|\mathrm{d}x| = \int_{\beta'} f(y)\,\frac{2|\mathrm{d}y|}{1+|y|^2},$$

where β is a circular arc joining P and Q, and $\beta' = p(\beta)$. Then β' is the straight line joining a and ∞, i.e. $\beta'(t) = a + t\,v$, where $v \in \mathbb{S}^{n-2} = \{x \in \mathbb{R}^{n-1} : |x| = 1\}$ and t goes from 0 to $+\infty$. Hence

$$d(P,Q) \leqslant \int_0^{+\infty} f(y)\,\frac{2\,\mathrm{d}t}{1+|y|^2}, \qquad y = a + t\,v.$$

Integrating with respect to $v \in \mathbb{S}^{n-2}$ and applying Fubini's theorem we conclude

$$d(P,Q) \leqslant \frac{2}{\sigma_{n-2}} \int_{\mathbb{S}^{n-2}} \left(\int_0^{+\infty} \frac{f(y)\,\mathrm{d}t}{1+|y|^2} \right) \mathrm{d}\sigma(v) = \frac{2}{\sigma_{n-2}} \int_Z \frac{f(y)\,\mathrm{d}V(y)}{|y-a|^{n-2}(1+|y|^2)},$$

where σ_{n-2} is the $n-2$ dimensional Lebesgue volume of $\mathbb{S}^{n-2}$. By Hölder's inequality we see that the last itegral on the right hand side is majorized by

$$\frac{A^{\frac{1}{n}}}{2} \left(\int_Z f^n(y)\,\frac{\mathrm{d}V(y)}{(1+|y|^2)^{n-1}} \right)^{\frac{1}{n}} = \frac{A^{\frac{1}{n}}}{2} \left(\int_S f^n(x)\,\mathrm{d}\sigma(x) \right)^{\frac{1}{n}},$$

where $\mathrm{d}V$ is the Lebesgue measure in $\mathbb{R}^{n-1}$ and $\mathrm{d}\sigma = \mathrm{d}\sigma_1$, and

$$A^{\frac{1}{n}} = \frac{4}{\sigma_{n-2}} \sup_{a \in Z} \left(\int_Z \frac{\mathrm{d}V(y)}{|y-a|^{\frac{n(n-2)}{n-1}} (1+|y|^2)^{\frac{1}{n-1}}} \right)^{\frac{n-1}{n}}.$$

We leave it to the reader to verify that A is finite.

Then we conclude that $\left(\int_\alpha f(x)\,|\mathrm{d}x|\right)^n \leqslant A \int_S f^n(x)\,\mathrm{d}\sigma(x)$. $\square$

Lemma 2.3. *Let β be a rectifiable curve in Ω whose one endpoint A_0 is in E_0 and the other A_1 in E_1. Let f be any metric in Ω. Then for each $a > 0$ there exists $b > 0$ such that, if we translate the curve β by a vector t, $|t| < b$ (notation β_t), then*

$$\int_{\beta_t} f\,|\mathrm{d}x| \geqslant L(\Gamma, f) - a,$$

where Γ is the family of all rectifiable Jordan arcs joining E_0 and E_1 inside Ω.

Remark. If we work with a ring domain, where E_0 and E_1 are boundary components, then, if there is part of the curve β_t outside Ω then β_t must intersect the corresponding boundary component, and we can choose the appropriate part of β_t which joins components (see [3] and [4]).

As we understand, in general we need an additional consideration because there is a possibility that β_t has a part outside Ω without intersection with E_0 or E_1.

Proof. Fix $a > 0$ and choose $\varepsilon > 0$ such that $\varepsilon = \frac{a^n \ln 2}{2^n A}$. There exists $b > 0$ such that

(i) the distance between E_0 and E_1 is greater than $4b$,

(ii) the diameter of each component of E_0 and E_1 is greater than $4b$,

(iii) $\iint_{|x-y|<2b} f^n \, \mathrm{d}V(x) < \varepsilon$ for each $y \in \overline{\Omega}$ (in fact, $\mu(A) = \int_A f^n \, \mathrm{d}V$ is an absolutely continuous measure with respect to the Lebesgue measure),

(iv) the outside boundary strip V'' is more than $4b$ thick.

By the Fubini theorem we have

$$\int_{b<|x-y|<2b} f^n(x) \, \mathrm{d}V(x) = \int_b^{2b} \frac{dr}{r} \int_{S_r} r f^n \, \mathrm{d}\sigma_r,$$

where S_r is the sphere of radius r with center at y.

So, then there exists $r_0 \in (b, 2b)$ such that

$$r_0 \int_{S_{r_0}} f^n \, \mathrm{d}\sigma_{r_0} \int_b^{2b} \frac{dr}{r} = r_0 \ln 2 \int_{S_{r_0}} f^n \, \mathrm{d}\sigma_{r_0} < \varepsilon,$$

i.e.

$$A \, r_0 \int_{S_{r_0}} f^n \, \mathrm{d}\sigma_{r_0} < \frac{A\varepsilon}{\ln 2} = \frac{a^n}{2^n}.$$

If we apply the above argument to $y = A_0$ then there exists $r_0 \in (b, 2b)$ such that

$$A r_0 \int_{S_{r_0}} f^n \, \mathrm{d}\sigma_{r_0} < \frac{a^n}{2^n}.$$

Let $B_0 \in S_{r_0} \cap \beta_t$ and $T_0 \in S_{r_0} \cap E_0$ (these intersections exist because the diameters of β_t and the components of E_0 are greater than $4b$). Then by Lemma 2.2 we can choose an arc α_0 on S_{r_0} joining T_0 and B_0 such that $L(\alpha_0, f) < \frac{a}{2}$.

In a similar way we can find a sphere S_{r_1} with center at A_1 and radius $r_1 \in (b, 2b)$, and choose a curve α_1 which joins the point B_1 of the curve β_t and the point T_1 on E_1, such that $L(\alpha, f) < \frac{a}{2}$.

From the arc $\alpha_0 + \beta_t + \alpha_1$ we choose a subarc γ which joins E_0 and E_1. Of course, γ is in Ω^* (which is a neighborhood of $\overline{\Omega}$). Every subarc of γ which is not in Ω can be mapped by A to be in Ω (we obtain a new arc γ'). Because $\gamma' \in \Gamma$ and by Lemma 2.1 we have

$$(1) \qquad \int_\gamma f \, |\mathrm{d}x| \geqslant \int_{\gamma'} f \, |\mathrm{d}x| \geqslant L(\Gamma, f)$$

and by (1) we conclude

$$\begin{aligned}\int_{\beta_t} f\,|\mathrm{d}x| &\geqslant \int_{\gamma} f\,|\mathrm{d}x| - \int_{\alpha_0} f\,|\mathrm{d}x| - \int_{\alpha_1} f\,|\mathrm{d}x| \\ &\geqslant L(\Gamma, f) - \frac{a}{2} - \frac{a}{2} = L(\Gamma, f) - a,\end{aligned}$$

which yields the desired conclusion. □

Proposition 2.1. *Under the above conditions we have*

$$\operatorname{mod}_\Omega(\Gamma) = d_\Omega(E_0, E_1)^{1-n} = \inf_g \frac{V(\Omega, g)}{L(\Omega, g)^n},$$

where the infimum is taken over all continuous metrics g in Ω.

Proof. Suppose that $0 < a < 1$ and f is any metric defined in Ω. Choose b as in Lemma 2.3.

Define g by

$$g(x) = \frac{1}{m(U_b)} \int_{U_b} f(x+y)\,\mathrm{d}V(y),$$

where $U_b = \{x\colon\ |x| < b\}$.

Then g is bounded and continuous. By Fubini's theorem for any $\beta \in \Gamma$ we have

$$\begin{aligned}(2)\qquad \int_\beta g\,|\mathrm{d}x| &= \int_\beta \biggl(\frac{1}{m(U_b)} \int_{U_b} f(x+y)\,\mathrm{d}V(y)\biggr)|\mathrm{d}x| \\ &= \frac{1}{m(U_b)} \int_{U_b} \biggl(\int_{\beta_y} f(x)|\mathrm{d}x|\biggr)\,\mathrm{d}V(y),\end{aligned}$$

where β_y denotes the translation of β through the vector y.

Now Lemma 2.3 implies that $\int_{\beta_y} f|\mathrm{d}x| \geqslant L(\Gamma, f) - a$ for each $|y| < b$, and we have by (2)

$$(3)\qquad L(\beta, g) = \int_\beta g\,|\mathrm{d}x| \geqslant L(\Gamma, f) - a,$$

and if we take the infimum in (3) over all such β, we obtain

$$(4)\qquad L(\Gamma, g) \geqslant L(\Gamma, f) - a.$$

Further, by Jensen's inequality we have

$$\begin{aligned}(5)\qquad V(\Omega, g) &= \int_\Omega g^n(x)\,\mathrm{d}V(x) \leqslant \frac{1}{m(U_b)} \int_{U_b} \int_\Omega f^n(x+y)\,\mathrm{d}V(x)\,\mathrm{d}V(y) \\ &\leqslant \int_{\Omega_b} f^n(x)\,\mathrm{d}V(x) = V(\Omega_b, f),\end{aligned}$$

where Ω_b is a b-neighborhood of $\overline{\Omega}$, and, by Lemma 2.1, $V(\Omega_b, f) \to V(\Omega, f)$ when $b \to 0$. By (4) and (5) we have

$$\frac{V(\Omega, g)}{L(\Gamma, g)^n} \leqslant \frac{V(\Omega_b, f)}{(L(\Gamma, f) - a)^n} \to \frac{V(\Omega, f)}{L(\Gamma, f)^n} \tag{6}$$

when $a \to 0$. From (6) we easily obtain the desired conclusion. □

Proposition 2.2. *Under the above conditions we have*

$$\inf_g \frac{V(\Omega, g)}{L(\Gamma, g)^n} = \inf_h \frac{V(\Omega, h)}{L(\Gamma, h)^n},$$

where g is any continuous metric and h is a metric from $C^\infty(\Omega)$.

P r o o f. Since g could be defined in a neighborhood Ω^* of $\overline{\Omega}$ then g can be aproximated by nonnegative C^∞-functions uniformly in the whole $\overline{\Omega}$. Let $h_k \rightrightarrows g$ in $\overline{\Omega}$ when $k \to \infty, h_k \in C^\infty(\Omega^*)$. Then

$$V(\Omega, h_k) \to V(\Omega, g),$$

and $L(\beta, h_k) \to L(\beta, g)$ for all $\beta \in \Gamma$, and also

$$L(\Gamma, h_k) \to L(\Gamma, g), \;\; k \to \infty.$$

Hence

$$\frac{V(\Omega, h_k)}{L(\Gamma, h_k)^n} \to \frac{V(\Omega, g)}{L(\Gamma, g)^n}, k \to \infty,$$

and we have the desired conclusion. □

Proposition 2.3. *Under the above conditions we have*

$$\inf_h \frac{V(\Omega, h)}{L(\Gamma, h)^n} = \inf_u \int_\Omega |\nabla u|^n \, \mathrm{d}V(x),$$

where h is any C^∞-metric and $u \in \mathrm{E}(\Omega, E_0, E_1)$.

P r o o f. We can define a function m by

$$m(x) = \inf_\beta \int_\beta h(y) \, |\mathrm{d}y|$$

and u by

$$u(x) = \min\Big(1, \frac{m(x)}{L(\Gamma, h)}\Big)$$

for each $x \in \overline{\Omega}$, where β is any Jordan arc joining x and E_0 inside Ω. Now, u satisfies the uniform Lipschitz condition and $u = 0$ on E_0 and $u = 1$ on E_1. Hence, $u \in \mathrm{E}(\Omega, E_0, E_1)$ and since $|\nabla u| \leqslant \frac{h}{L(\Gamma,h)}$ a.e. in Ω we have

$$\int_\Omega |\nabla u|^n \,\mathrm{d}V(x) \leqslant \frac{1}{(L(\Gamma,h))^n} \int_\Omega h^n \,\mathrm{d}V(x) = \frac{V(\Omega,h)}{L(\Gamma,h)^n}.$$

We have proved the proposition. □

Proposition 2.4. *Under the above conditions we have*

$$C[\Omega, E_0, E_1] = \inf_u \int_\Omega |\nabla u|^n \,\mathrm{d}V(x), \tag{7}$$

where the infimum is taken over all $u \in \mathrm{E}(\Omega, E_0, E_1)$.

P r o o f. For $u \in \mathrm{E}(\Omega, E_0, E_1)$ one can conclude that u can be extended to a neighborhood Ω^* of $\overline{\Omega}$ such that u remains continuous and ACL. We may assume that $|\nabla u|$ is L^n-integrable over Ω^*. Next fix $0 < a < \frac{1}{2}$ and let

$$v = \begin{cases} 0, & \text{if } u < a \\ \dfrac{u-a}{1-2a}, & \text{if } a \leqslant u \leqslant 1-a \qquad \text{on } \overline{\Omega}. \\ 1, & \text{if } 1-a < u \end{cases} \tag{8}$$

The set where $a \leqslant u \leqslant 1-a$ is a bounded subset of $\mathbb{R}^n$ and lies at a distance b from $E_0 \cup E_1$. Let

$$\omega(x) = \frac{1}{m(U_c)} \int_{U_c} v(x+y) \,\mathrm{d}V(y),$$

where $c < b$.

This function is continuously differentiable in Ω and has boundary values 0 on E_0 and 1 on E_1. From (8) we see that v is ACL everywhere and by Hölder's inequality we obtain that $|\nabla v|$ is L^n-integrable over each compact set. Hence, we can apply Fubini's theorem to conclude that

$$\nabla\omega(x) = \frac{1}{m(U_c)} \int_{U_c} \nabla v(x+y) \,\mathrm{d}V(y)$$

for each $x \in \Omega$. Then applying Jensen's inequality we obtain

$$\int_\Omega |\nabla\omega(x)|^n \,\mathrm{d}V(x) \leqslant \frac{1}{m(U_c)} \int_{U_c} \int_\Omega |\nabla v(x+y)|^n \,\mathrm{d}V(x) \,\mathrm{d}V(y).$$

The inner integral on the right hand side is majorized by

$$\int_{\Omega_c} |\nabla v(x)|^n \,\mathrm{d}V(x) \leqslant \frac{1}{(1-2a)^n} \int_{\Omega_c} |\nabla u(x)|^n \,\mathrm{d}V(x)$$

for each y in U_c. Hence

$$\int_{\Omega} |\nabla \omega|^n \,\mathrm{d}V(x) \leqslant \frac{1}{(1-2a)^n} \int_{\Omega_c} |\nabla u|^n \,\mathrm{d}V(x)$$

and

$$C[\Omega, E_0, E_1] \leqslant \frac{1}{(1-2a)^n} \int_{\Omega_c} |\nabla u|^n \,\mathrm{d}V(x).$$

Letting $a \to 0$ we have

$$C[\Omega, E_0, E_1] \leqslant \int_{\Omega} |\nabla u|^n \,\mathrm{d}V(x). \tag{9}$$

Because the infimum on the right hand side of (7) is over a wider class of functions than on the left hand side we have the inequality

$$C[\Omega, E_0, E_1] \geqslant \inf_u \int_{\Omega} |\nabla u|^n \,\mathrm{d}V(x). \tag{10}$$

By (9) and (10) we have the desired conclusion. □

Theorem 2.1. *If Ω is a bounded domain whose boundary consists of a finite number of C^1 hypersurfaces, and if E_0 and E_1 are disjoint subsets of the boundary of Ω consisting of a finite number of closed hypersurfaces, then we have*

$$\operatorname{mod}_{\Omega}(\Gamma) = \inf_f \frac{V(\Omega, f)}{L(\Gamma, f)^n} = C[\Omega, E_0, E_1], \tag{11}$$

where f is any metric in Ω and Γ is the family of Jordan arcs joining E_0 and E_1 inside Ω.

P r o o f. It follows by Propositions 2.1, 2.2, 2.3 and 2.4. □

The case $n = 2$ of the above Theorem enables us to give a short proof of Theorem 1.1. In fact, the proof immediately follows from Theorem 1.3 [6], which gives a solution of a mixed Dirichlet-Neuman problem.

The proof of Theorem 1.3 in Courant's book [6] is based on using minimizing sequences. We believe that we can use minimizing sequences as Gehring in [2] to show the existence of the extremal admissible function $u \in \mathrm{E}(\Omega, E_0, E_1)$ such that

$$C[\Omega, E_0, E_1] = \int_{\Omega} |\nabla u|^n \,\mathrm{d}V.$$

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